JOURNAL OF PURE AND APPLIED ALGEBRA - Volume 1, No. 3 (1971) pp. 253-273.

SUBGROUPS OF FINITE DIMENSIONAL TOPOLOGICAL GROUPS

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Received 3 February 1971

H-spaces which have the homotopy type of a finite CW-complex and, in particular, topological groups with that property share many of the homotopy theoretic properties of Lie groups. In particular, such spaces satisfy Poincaré duality [14, 7.9] and have rational cohomology rings which are exterior algebras on generators of odd degree [19]. Recently, the study of such H-spaces has been enriched by the construction, using a technique of A. Zabrodsky, of many examples which are not homotopy equivalent to any Lie group [34, 30, 18, 15].

Two sets of phenomena which have proven important in the theory of Lie groups are the maximal torus with its associated Weyl group and the properties of the cohomology ring of a homogeneous space. It is therefore interesting to know to what extent these phenomena generalize to the homotopy category of topological groups with the homotopy type of a finite complex (to be called FD-groups below). The generalization of these phenomena depends on giving a definition of subgroup which makes sense in the homotopy category. In §1 we propose such a definition which essentially requires that the quotient space of a group by a subgroup have the homotopy type of a finite complex. We will prove in §2 that quotient spaces are Poincaré complexes and that their cohomology rings satisfy various theorems of Borel et al., when the subgroup is of maximal rank. In §3 we examine the case of a subtorus and define an analogue of the Weyl group. We will show that this group is isomorphic to the classical Weyl group in case the FD-group and subtorus are a Lie group and a maximal subtorus in the classical sense.

In the remainder of the paper, we will provide examples of groups and subgroups, and, in particular, we will provide an example of a group which contains no torus of maximal rank. For the latter example we will use an exotic multiplication on S^3 first constructed by Slifker [28]. We will give a much simplified construction of these multiplications using Zabrodsky's homotopy mixing construction [34].

The author would like to thank Al Vasquez for a very useful discussion leading to the proof of Theorem 3.6. I would also like to thank Larry Smith and Frank Quinn for suggesting the central idea in the proof of Theorem 2.1, which is analogous to a theorem in the latters thesis (Princeton 1970).

[1] The author was partially supported by NSF-grant GP-20552 during the preparation of this work

§ 1. Subgroups

Results of Stasheff [29] et al., show that for homotopy theoretic purposes we may regard a topological group to be the loop space on some space and a homomorphism to be induced by a homotopy class of continuous mappings of spaces. For if G is a topological group, B_G its classifying space [25], then there is an ΩB_G fitted with a natural group structure and a homomorphism $\Omega B_G \to G$ which is an A_∞ equivalence in the sense of Stasheff. We will thus call two groups equivalent if they have homotopy equivalent classifying spaces.

The group structure on a loop space is technically easiest to construct in the appropriate simplicial category. If X is a simplicial set with one vertex, there is a free simplicial group GX and a principal fibration

$$GX \to PX \to X$$

with PX contractible [30, 22]. There is also for a simplicial group G a classifying fibration

$$G \to WG \to \overline{W}G$$

[26]. In the category of spaces, ΩX may be given a group structure by using a Moore loop space with appropriate reductions to insure the existence of inverses. Care must be taken to insure that all spaces, in particular product spaces, be given a compactly generated topology and are of the homotopy type of a CW-complex. The author is prepared to take the responsibility for details in the simplicial category and therefore the notation GX will be used for the loop complex of X. The more familiar notation B_G will be used for the classifying complex. Whenever an obvious space such as a Lie group is mentioned, the singular complex of that space will be meant. It will not be necessary for us to explicitly consider technical details in the simplicial categories.

We shall mean by an *FD-group structure* a simplicial set X with a single vertex such that GX has the homotopy type of a finite complex. A *homomorphism* is a simplicial map $X \to Y$ of FD-group structures. Two homomorphisms are equivalent if they are homotopic and two group structures are equivalent if they are homotopy equivalent. If K is a finite complex, we shall mean by a *group structure on* K an FD-group structure X such that $GX \simeq K$. If X is an equivalence class of FD-group structures we will call $H = GX$ an *FD-group*. Now, given a simplicial group H, we have a group structure B_H functorially determined. This structure is unique up to equivalence since there is a natural homotopy equivalence $X \simeq B_{GX}$. So if H is an FD-group, we denote by B_H the FD-group structure determining it. A similar convention will be applied to homomorphisms.

Let $H \xrightarrow{f} G$ be a homomorphism of FD-groups. By replacing B_G by a homotopy equivalent complex $\widetilde{B}_G$ (mapping cylinder) we may replace the map B_f by an equivalent inclusion. Thus we may replace the homomorphism f by an equivalent inclusion of simplicial groups $GB_H \to G\widetilde{B}_G$. We then have a principal fibre bundle [8]

$$GB_H \to G\widetilde{B}_G \to G\widetilde{B}_G/GB_H.$$

The homotopy type of the base of this fibre bundle is easily seen to be determined by the equivalence class of f. We denote the base by $G//f$. We now propose

1.1. Definition. A homomorphism $f: H \to G$ of FD-groups is an *inclusion of FD-groups* if $G//f$ has the homotopy type of a finite complex. In that case we denote $G//f$ by G/H and the principal fibration above by

$$H \xrightarrow{f} G \to G/H \quad .$$

An alternate definition could be obtained as follows. Replace B_H by an equivalent complex $\tilde{B}_H$ and B_f by an equivalent map $\tilde{B}_f$ so that $\tilde{B}_f$ is a Kan fibration with fibre F. Then $F \simeq G//f$. So

1.2. Proposition. *$f: H \to G$ is an inclusion of FD-groups iff the fibre of B_f (made into a fibration) has the homotopy type of a finite complex.*

Our notion of inclusion is a generalization of the classical notion for Lie groups in the following sense. Let $H \to G$ be a pair of compact Lie groups. Then G/H is a compact manifold and is homotopy equivalent to the fibre of the inclusion of classifying spaces $B_H \to B_G$; thus

1.3. Proposition. *If $H \to G$ is an inclusion of Lie groups it is an inclusion of FD-groups.*

We have the following transitivity property of FD-inclusion.

1.4. Proposition. *Let $K \subsetneq H$ and $H \subsetneq G$ be FD-inclusions. Then the composition $K \subsetneq G$ is an FD-inclusion.*

Proof. We have fibrations

$$F_0 \to B_H \to B_G$$
$$F_1 \to B_K \to B_H$$

with F_0 and F_1 having the homotopy type of finite complexes. Let F_{01} be the fibre of the composition $B_K \to B_H \to B_G$. Then there is a fibration

$$F_1 \to F_{01} \to F_0 \, .$$

By [23; Th. 1] the Wall obstruction to finiteness of F_{01} [33] vanishes.

Now suppose $F \to X \xrightarrow{f} Y$ is a fibration. Then the sequence of simplicial groups

$$GF \to GX \xrightarrow{Gf} GY$$

is not an exact sequence. But if $\tilde{G}F = \operatorname{Ker} Gf$, then there is an inclusion of free simplicial groups $GF \subset \operatorname{Ker} Gf$ which induces isomorphisms of homotopy groups. Thus $B_{GF} \simeq B_{\tilde{G}F} \simeq F$. So we have an exact sequence of groups

$$\tilde{G}F \to GX \to GY$$

with $\tilde{G}F$ equivalent to GF. Thus we may make

1.5. Definition. A sequence of FD-groups $K \to H \to G$ is an *extension* if it is equivalent to an exact sequence of groups, i.e., if the sequence $B_K \to B_H \to B_G$ is a quasifibration [17]. An FD-inclusion $K \to H$ is *normal* if it is equivalent to the fibre of an extension.

Let K and G be FD-groups. We wish to classify all extensions $K \to H \to G$. This is equivalent to classifying fibre homotopy equivalent fibrations

$$B_K \to B_H \to B_G.$$

Let $H(B_K)$ be the complex of homotopy equivalences of B_K (see Appendix). Then fibre homotopy classes of fibrations $B_K \to X \to B_G$ are in one to one correspondence with the elements of $[B_G, H(B_K)]$ (see Appendix). Thus

1.6. Proposition. *Let K and G be FD-groups. The equivalence classes of extensions of FD-groups $K \to H \to G$ correspond one for one to elements of $[B_G, H(B_K)]$.*

Proof. We have yet to show that if

$$B_K \to X \to B_G$$

is a fibration then X is an FD-group structure. But there is a fibration $F \to GX \to GB_G$, $F \simeq GB_K$ with GB_K and GB_G having the homotopy type of finite complexes. Thus, again by [23, Th. 1], GX has the homotopy type of a finite complex.

§ 2. FD-homogeneous spaces

We will show in this section that quotients G/H of a pair of FD-groups $H \subseteq G$ share many of the homotopy theoretic properties of homogeneous spaces of Lie groups.

2.1. Theorem. *Let G and H be connected FD-groups, H an FD-subgroup of G. Then G/H is a Poincaré complex.*

Proof. Let $\dim H = m$, $\dim G = n$, $\dim G/H = l$. Consider the principal fibration

$$H \to G \to G/H.$$

Since H and G are connected groups, $\pi_1 G/H$ acts trivially on all homology groups in sight. Let $\{E^r\}$ and $\{E_r\}$ be the homology and cohomology Serre spectral sequences of the fibration above with coefficients in the commutative ring Λ. Since $\dim H = m$, $\dim G/H = l$, it is clear from the spectral sequence $\{E^r\}$ that $\dim G = l + m$. In fact, since $H_m(H; Z) = Z$, $H_n(G; Z) = Z$, and no nontrivial differentials enter or leave $E^r_{l,m}$, we have

$$H_l(G/H;Z) = E^2_{l,m} = H_n(G;Z) = Z\,.$$

The basic idea of our proof is this. We want to construct a map of spectral sequences

$$\{f_r : E_r^{p,q} \to E^r_{l-p,m-q}\}$$

such that

(i) $f_2 : E_2^{0,q} \to E^2_{l,m-q} \approx H_{m-q}(H;\Lambda)$ is the duality map $H^q(H;\Lambda) \to H_{m-q}(H;\Lambda)$.

(ii) $f_2 : E_2^{p,0} \to E^2_{l-p,m} \approx H_{l-p}(G/H;\Lambda)$ is cap product with the top class.

(iii) The duality map $H^i(G;\Lambda) \to H_{n-i}(G;\Lambda)$ preserves filtrations from the spectral sequence, and $f_\infty : E_\infty \to E^\infty$ is the graded map associated to the duality map.

If these conditions are satisfied, then with appropriate reindexing of $\{F^r\}$, the maps f_r satisfy the conditions of the Zeeman comparison theorem [35]. Thus the cap product $H^p(G/H;\Lambda) \to H_{l-p}(G/H;\Lambda)$ will be an isomorphism. To carry out this program without going to the trouble of investigating cap products in $\{E_r\}$ we proceed as follows.

First we suppose Λ to be a field k. For each finite complex X we may identify $H_*(X;k)$ with $(H^*(X;k))^*$. Also we may identify $E^r_{p,q}$ with $(E_r^{p,q})^*$. The filtration of $H_*(G;k)$ from the spectral sequence may now be identified as

$$F^r H_*(G;k) = \{f \in (H^*(G;k))^* \mid f(F_{r+1}H^*(G;k)) = 0\}\,.$$

Duality arises as follows. We have a graded symmetric bilinear form $\langle\ ,\ \rangle$: $H^*(G,k) \otimes H^*(G,k) \to k$ given by

$$\langle x,y\rangle = \begin{cases} xy, & \dim x + \dim y = n\,,\\ 0 & \text{otherwise.}\end{cases}$$

The duality map of G is $f : H^*(G;k) \to H_{n-*}(G;k)$ given by

$$f(x)(y) = \langle x,y\rangle\,.$$

This map is an isomorphism. Now in $H^*(G;k)$,

$$F_p \cdot F_q = 0\,, \quad p+q > l$$

since $E^2_{s,*} = 0$, $s > l$. Therefore, $f(F_p) \subseteq F^{l-p}$. Similarly, define a form $\langle\ ,\ \rangle_r$ on E_r by

$$\langle x,y\rangle_r = \begin{cases} x\cdot y\,, & \operatorname{bideg} x + \operatorname{bideg} y = (l,m),\\ 0\,, & \text{otherwise.}\end{cases}$$

We have f_r given by

$$f_r(x)(y) = \langle x, y\rangle_r \ .$$

The maps f_r satisfy the conditions (i), (ii), (iii) above. All that remains is to show $f_r \circ d_r = d^r \circ f_r$. But if bideg x + bideg $d_r y = (l, m)$, then $f_r(x)(d_r y) = x \cdot d_r y = \pm d_r x \cdot y = f_r(d_r x)(y)$ since d_r is a derivation. We have thus proven the theorem.

We will now give detailed results for $H^*(G/H; k)$ when H is of maximal rank. Recall [19] that if G is any connected FD-H space, then

$$H^*(G; Q) = \Lambda_Q[x_{2r_1-1}, \dots, x_{2r_n-1}] \ ,$$

an exterior algebra on n generators of odd degree. The number n is an invariant of G and we call it rank G. If H is an FD-subgroup of G, we say H is of *maximal rank* if rank H = rank G.

Let k be a field, $p = \operatorname{char} k$. Suppose $p = 0$, or $H^*(G; Z)$ has no p-torsion; then

$$H^*(G; k) = \Lambda_k[x_{2r_1-1}, \dots, x_{2r_n-1}] \ ,$$

and, by a standard Serre spectral sequence argument,

$$H^*(B_G; k) = k[y_{2r_1}, \dots, y_{2r_n}] \ ,$$

where the generators x_{2r_i-1} transgress to y_{2r_i}.

Many of the classical theorems on cohomology of homogeneous spaces (see e.g. [12]) are included in

2.2. Theorem (Borel [11]). *Let H be a connected FD-subgroup of the connected FD-group G, k a field of characteristic p. Suppose*

(a) *H and G have no p-torsion or $p = 0$,*

(b) *H is of maximal rank.*

Then, if $f: B_H \to B_G$ is the inclusion,

(i) *$H^*(B_G; k) \xrightarrow{f^*} H^*(B_H; k)$ is monic,*

(ii) *$H^*(G/H; k) = H^*(B_H; k)//H^*(B_G; k)$ as an algebra* (i.e. $H^*(G/H; k) = H^*(B_H; k) \otimes_{H^*(B_G; k)} k$),

(iii) *$H^*(B_H; k) \approx H^*(G/H; k) \otimes_k H^*(B_G; k)$ as a $H^*(B_G; k)$ module, i.e. $H^*(B_H; k)$ is a free $H^*(B_G; k)$ module of dimension equal to $\chi(G/H)$, where χ is the Euler characteristic.*

It is an open question whether the attempted generalization in the Lie case by Baum [9, 10] to subgroups of less than maximal rank holds for FD-groups. The result for Lie groups has been announced by P. May.

Proof of 2.2. The proof by Baum [10] goes through without change. We outline the proof here since we wish to draw some corollaries from the proof. The essential ingredients are the following two lemmas. Let $\Lambda = k[x_1, \ldots, x_n]$, $\deg x_i > 0$, and let $I \subset \Lambda$ be an ideal, $I = (y_1, \ldots, y_m)$, where the y_i are homogeneous elements of Λ and are an irredundant set of generators of I. Then [10]

2.3. Lemma. *$\dim_k \Lambda/I$ is finite iff $m \geqslant n$.*

2.4. Lemma. *If $\dim_k \Lambda/I$ is finite, I is a Borel ideal iff $m = n$.*

The theorem now follows from these lemmas and a spectral sequence argument on the fibration

$$G/H \to B_H \to B_G.$$

From 2.3 we have immediately,

2.5. Corollary. *If H is an FD-subgroup of G, then rank $H \leqslant$ rank G.*

§3. Tori and the Weyl group

Let T be a connected homotopy Abelian FD-group. Then by a theorem of Hubbuck [20], T has the homotopy type of a product of circles. But then B_T is a product of $K(\mathbf{Z}, 2)$'s and is unique up to homotopy type. Therefore

3.1. Proposition. *There is a unique torus T^n with* rank $T^n = \dim T^n = n$.

Corollary 2.5 becomes

3.2. Proposition. *If G is an FD-group and T a subtorus, then* rank $T \leqslant$ rank G.

An FD-group need not have a torus of maximal rank. In §3 and §4 we will provide an example.

Let $B_T \xrightarrow{f} B_G$ be a subtorus. We have the following analogue of the Weyl group of a Lie group.

3.3. Definition. The Weyl group of G with respect to the inclusion of a subtorus $f\colon B_T \to B_G$ is

$$W(G, f) = \{\alpha \in [B_T, B_T] \mid \alpha \text{ is a homotopy equivalence and } f \circ \alpha \sim f\}.$$

Note that since B_T is an Eilenberg-MacLane space, $[B_T, B_T]$ is isomorphic to $\mathrm{Hom}(\mathbf{Z}^n, \mathbf{Z}^n)$, so

(3.4) $\qquad W(G, f) \subseteq \mathrm{Gl}(n, \mathbf{Z}),\quad \text{rank } T = n.$

3.5. Remark. We may generalize W as follows. Let $H(G, f)$ be the complex of all homotopy equivalences $\alpha: B_T \to B_T$ (see Appendix) such that $f \circ \alpha \sim f$. Let

$$W_*(G, f) = \pi_* H(G, f).$$

Clearly $W_0(G, f) = W(G, f)$. Perhaps these groups will provide useful information about G if they can be calculated. It is plausible that $W_i(G, f) = 0$, $i > 0$, if G is a Lie group and f the classifying map of a Lie maximal torus.

We are justified in calling W a Weyl group since

3.6. Theorem. *Let G be a connected Lie group, T^n a maximal torus (in the Lie sense), and W the Weyl group of G. Then*

$$W \approx W(G, f)$$

where f is the classifying map of the inclusion $T \to G$.

Proof. Let $\alpha \in W$. Then α induces a nontrivial automorphism $H^1(T^n, Z) \to H^1(T^n, Z)$, so α may be considered to be an element of $\mathrm{Gl}(n, Z)$. To see that $\alpha \in W(G, f)$, recall that α is induced by an inner automorphism $a^{-1}(\ \ )a$ sending T into T. Let $h: a \sim 1$ be a path from a to 1. Then define $H: T \times I \to G$ by

$$H(x, t) = h(t)^{-1} x\, h(t).$$

H is a continuous deformation by homomorphisms of $a^{-1}(\ \ )a$ to the inclusion $T^n \to G$. Thus $a^{-1}(\ \ )a$ and the inclusion induce homotopic maps $B_T \to B_G$. Thus $W \subseteq W(G, f)$. We must now show $W(G, f) \subseteq W$. Let $\Lambda = H^*(B_T; Q)$. Then $\Lambda = Q[x_1, ..., x_n]$, $\dim x_i = 2$. Now $H^*(B_G; Q)$ may be considered to be a subalgebra Γ of Λ, and $\Gamma = \Lambda^W$, the fixed subalgebra under the action of W. Clearly Γ is also fixed under the action of $W(G, f)$. To show $W(G, f) \subseteq W$ we will use some elementary Galois theory.

Let K be the quotient field of Λ, K^W the fixed subfield under the action of W, and $K_0 \subseteq K^W \subseteq K$ the quotient field of Γ. Now $K : K^W$ is a normal extension [5, Cor. to Th. 14]; so it suffices to show K^W is fixed under $W(G, f)$. But $W(G, f)$ leaves K_0 fixed so it suffices to show $K_0 = K^W$. Now $[K : K^W] = |W|$, so it suffices to show $[K : K_0] = |W|$. We recall from Lie theory that Λ is a free Γ module and $\dim_\Gamma \Lambda = |W|$ [11, 27.1]. Since Λ is a finite Γ module, Λ is integral over Γ [32, pp. 74–76]. Furthermore, since every rational function which is integral over the polynomial ring Λ is in Λ, Λ is the integral closure of Γ in K. But then [32, p. 78] any $x \in K$ may be written $x = a/b$ where $a \in \Lambda$ and $b \in \Gamma$. Thus $[K: K_0] = \dim_\Gamma \Lambda = |W|$ and $K_0 = K^W$.

3.7. Remark. We see from the above proof that the analogue for FD-groups of the classical theorem that any two maximal tori of a Lie group are conjugate would be that any two inclusions $f, g: B_T \to B_G$ of a torus T of maximal rank be homotopic up to a self equivalence of B_T. We do not as yet have such a theorem. Indeed, if G is a Lie group, we do not know whether an inclusion of a torus is homotopic to the classifying map of a Lie inclusion. Thus the Weyl group defined above must *a priori* be taken to depend on the inclusion f.

§4. Tori in groups on S^3

We here undertake to provide an example of an FD-group containing no non-trivial torus. We will use an exotic structure on S^3. To distinguish between group structures on S^3 and to refute the existence of subtori in appropriate cases, we will use Adams operations in KO-theory.

Let B be a group structure on S^3, i.e., $GB \simeq S^3$. By using the Serre spectral sequence of the fibration

$$GB \to PB \to B$$

we may show that

$$H^*(B;Z) = Z[x_4],$$

$\dim x_4 = 4$. We want to calculate the KO-theory of B. For a systematic account of KO-theory see [13], or [3]. We need the following facts:

$$\mathrm{KO}^i(pt) = \begin{cases} Z, & i \equiv 0 \text{ or } -4 \pmod 8 \\ Z_2, & i \equiv -1 \text{ or } -2 \pmod 8 \\ 0, & \text{otherwise;} \end{cases}$$

$$\mathrm{KU}^i(pt) = \begin{cases} Z, & i \equiv 0 \pmod 2 \\ 0, & \text{otherwise.} \end{cases}$$

We have a generator η of $\mathrm{KU}^{-2}(pt)$ so that $\mathrm{KU}^*(pt) = Z[\eta, \eta^{-1}]$, and generators $\eta_1, \eta_2, \eta_4, \eta_8, \eta_i \in \mathrm{KO}^{-i}(pt)$ so that $\mathrm{KO}^*(pt)$ is generated by $\eta_1, \eta_4, \eta_8, \eta_8^{-1}$ as an algebra with relations $2\eta_1 = 0, \eta_1^2 = \eta_2, \eta_1\eta_2 = 0, \eta_1\eta_4 = 0, \eta_4^2 = 4\eta_8$. Under the complexification homomorphism c: $\mathrm{KO}^*(X) \to \mathrm{KU}^*(X)$, $c\eta_4 = 2\eta^2$, $c\eta_8 = \eta^4$.

We may now use the Atiyah-Hirzebruch spectral sequence for B [3, 7] to calculate $\mathrm{KO}^*(B)$ and $\mathrm{KU}^*(B)$.

4.1. Lemma. $\mathrm{KO}^*(B) = \mathrm{KO}^*(pt)[[x]]$, $x \in \mathrm{KO}^4(B)$, *and x represents $x_4 \in E_2^{4,0}(B)$ in the Atiyah-Hirzebruch spectral sequence.* $\mathrm{KU}^*(B) = \mathrm{KU}^*[[\bar{x}]]$, $\bar{x} \in \mathrm{KU}^4(B)$, $\bar{x}$ *representing x_4, and $\bar{x}$ may be chosen so that $cx = \bar{x}$.*

Proof. The spectral sequences collapses for dimensional reasons.

In order to distinguish homotopy types of the spaces B we are going to use the Adams operation Ψ^2 [3] applied to $\eta_4 x \in \mathrm{KO}(B)$. To do this we must know how this operation depends on the choice of the generator x. We will only be concerned with this operation evaluated on $\eta_8 x^2$ so we will do our computations in $\mathrm{KO}^*/\mathrm{KO}_9^*$.

Note that an additive basis for $\mathrm{KO}^4(B)/\mathrm{KO}_9^4(B)$ is $\{x, \eta_4 x^2\}$. If x' is another representative of x_4, then $x' = x + e\eta_4 x^2$. Put $y = \eta_4 x$, $y' = \eta_4 x'$, $z = \eta_8 x^2$, $\bar{y} = \eta^2 \bar{x}$. Then $y' = \eta_4 x' = \eta_4 x + e\eta_4^2 x^2 = y + 4ez$. Now since $y \in \mathrm{KO}_4$, $\Psi^2 y = 4y + a'z$ [3, 5.2]. We assert a' is divisible by two. For $cy = 2\bar{y} \in \mathrm{KU}(B)$ and $cz = \bar{y}^2$, so $c\Psi^2 y = 2(\Psi^2 \bar{y}) = 8\bar{y} + 2a\bar{y}^2 = c(4y + 2az)$. Thus

$$\Psi^2 y = 4y + 2az \pmod{\mathrm{KO}_9}.$$

We must determine how a depends on the choice of x. Calculate modulo $\mathrm{KO}_9(B)$ to get

$$\begin{aligned}
\Psi^2 y' &= \Psi^2(y + 4ez) \\
&= 4y + 64ez + 2az \\
&= 4y' + (2a + 48e)z \\
&= 4y' + 2(a + 24e)z .
\end{aligned}$$

Thus a is well determined (mod 24). The choice of x_4 was unique up to sign. Also $\Psi^2(-y) = 4(-y) - 2az$. Denote a by $a(B)$. We have proved

4.2. Lemma. *If $B \simeq B'$, then $a(B) \equiv \pm a(B')$ (mod 24)*

We will prove in the next section that

4.3. Theorem. *There exist FD-group structures B on S^3 such that*

$$a(B) \equiv \pm 1, \pm 5, \pm 7, \quad \text{or} \quad \pm 11$$

(mod 24).

4.4. Remark. Note that $\mathrm{KO}(B)/\mathrm{KO}_9(B)$ is just the KO-theory of the projective plane of the multiplication on S^2 determined by B. The invariant $\pm a(B)$ distinguishes the four H-classes of multiplications on S^3 which may be made associative [28].

The main result of this section is

4.5. Theorem. *If B is a group structure on S^3 with $a(B) = \pm 5$ or ± 7, then B contains no non-trivial torus. There is a B with $a(B) = \pm 1$ which contains a circle.*

Proof. The B with $a(B) = \pm 1$ maybe taken to be HP^∞ by 4.6 below. The case $a(B) = \pm 11$ is as yet undetermined. Let B be a group structure on S^3, x_4 the generator of $H^4(B; \mathbf{Z})$ and $f\colon CP^\infty \to B$ a subtorus of rank one. Now $H^*(CP^\infty; \mathbf{Z}) = \mathbf{Z}[t]$, $\dim t = 2$. An immediate consequence of Theorem 2.2 is that $f^*(x_4) = \pm t^2$. When $a(B) = \pm 5$ or ± 7, we will show f does not exist by showing that $f^!$ cannot commute with Ψ^2. We get no more information from $\mathrm{KO}(CP^\infty)$ than from $\mathrm{KU}(CP^\infty)$ so we will calculate in KU-theory.

Recall that $KU(CP^\infty) = Z[[\xi]]$, where $\xi + 1$ is the canonical line bundle, and ξ represents t in the Atiyah-Hirzebruch spectral sequence. Furthermore

$$\Psi^k \xi = (\xi + 1)^k - 1$$

[3]. Chose x_4 so that $f^*(x_4) = t^2$. We have $KU(B) = Z[[\bar{y}]]$, y representing x_4. Therefore

$$f^! \bar{y} \equiv \xi^2 + e_3\xi^3 + e_4\xi^4 \quad (\mathrm{mod}\, KU_9).$$

Furthermore

$$\Psi^2 \bar{y} \equiv 4\bar{y} + a\bar{y}^2 \quad (\mathrm{mod}\, KU_9).$$

Calculate $(\mathrm{mod}\, KU_9)$:

$$\begin{aligned} \Psi^2 f^! \bar{y} &\equiv \Psi^2(\xi^2 + e_3\xi^3 + e_4\xi^4) \\ &\equiv 4\xi^2 + (4 + 8e_3)\xi^3 + (1 + 12e_3 + 16e_4)\xi^4, \end{aligned}$$

$$\begin{aligned} f^! \Psi^2 y &\equiv f^!(4\bar{y} + a\bar{y}^2) \\ &\equiv 4\xi^2 + 4e_3\xi^3 + (a + 4e_4)\xi^4. \end{aligned}$$

Equating coefficients gives $e_3 = -1$ and

$$a = (1 + 12(e_4 - 1)).$$

Therefore

$$a \equiv 1 \ (\mathrm{mod}\ 12).$$

The theorem is thus proved.

4.6. Lemma. $a(HP^\infty) = 1$.

Proof. If $B = HP^\infty$ in Lemma 4.1, the generator x may be chosen so that $\eta_8 x + 1_H = \lambda$, the canonical quaternionic line bundle [31, p. 250]. Let $f\colon CP^\infty \to HP^\infty$ be the inclusion and let $\zeta = \xi + 1$ be the canonical line bundle over CP^∞. Then

$$f^! \lambda = \zeta \oplus \bar{\zeta}$$

as a complex bundle [31, p. 251]. Therefore as a complex virtual bundle

$$\begin{aligned} \eta^{-2} c f^! \eta_8 x &= 2 - \zeta \oplus \bar{\zeta} \\ &= 1 - \xi - \frac{1}{1+\xi} \\ &= \xi^2/(1+\xi) \end{aligned}$$

since $\bar{\zeta} = 1/\zeta$ [3, 5.1]. Therefore

$$\Psi^2\eta^{-2}cf^!\eta_8 x = (4\xi^2 + 4\xi^3 + \xi^4)/(1+\xi)^2$$
$$= 4\xi^2/(1+\xi) + \xi^4/(1+\xi)^2.$$

Consequently,

$$\Psi^2\eta_4 x = 4\eta_4 x + 2\eta_8 x^2$$

since $f^!$ and c are monic [4], [1, §2].

§ 5. Multiplications on S^3

We will now provide the multiplications of Theorem 4.3. Suitable examples were first constructed by Slifker [28] using other methods. We give here a much simplified construction using the Zabrodsky mixing procedure [34]. We will use this construction in the next section to find sub three-spheres in an exotic FD-group.

Let

$$f_i^!: HP^\infty \to K(Z,4), \quad i = 0, 1,$$

be maps such that

(i) $f_i^!$ represents n_i times a chosen generator u_4 of $H^4(HP^\infty; Z)$, where

(ii) n_0 is odd, and

(iii) n_1 is a power of two.

By a theorem of Zabrodsky [34, 1.2], we may factor $f_i^!$ by

$$HP^\infty \xrightarrow{f_i} M_i \xrightarrow{f_i''} K(Z,4),$$

where

(i) f_i'' is a fibre map, $i = 0, 1$,
(ii) f_0 and f_1'' are mod 2 equivalences [27],
(iii) f_0'' and f_1 are mod 3, 5, 7, ... equivalences,
(iv) all maps f_i, f_i'' are rational equivalences,
(v) $f_i^*: H^4(M_i; Z) \to H^4(HP^\infty; Z)$ has degree n_i, $i = 0, 1$.

Let B be the fibre product of f_0'' and f_1''. We have a commuting diagram

(5.1)
$$\begin{tikzcd}
HP^\infty \arrow[r, "f_0", "2\sim"'] & M_0 \arrow[rr, "f_0''", "3,5,7,\ldots\sim"'] & & K(Z,4) \\
& B \arrow[u, "\rho_0"] \arrow[rr, "\rho_1"] & & M_1 \arrow[u, "f_1''"', "2\sim"] \\
& & & HP^\infty \arrow[u, "f_1", "3,5,7,\ldots\sim"']
\end{tikzcd}$$

It is clear from a Serre spectral sequence argument that ρ_0 is a 2-equivalence and ρ_1 is a 3,5,7, ...-equivalence. Thus

5.2. Lemma. $H^*(B;Z) = Z[x_4]$, $\dim x_4 = 4$, *and* $GB \simeq S^3$.

5.3. Proposition. $\pm a(B) \equiv 9n_0 - 8n_1 \pmod{24}$.

That this implies Theorem 4.3 is evident from the following table

$n_0 \pmod 8$	$n_1 \pmod 3$	$\pm a(B) \pmod{24}$
1	1	± 1
1	-1	± 7
3	1	± 5
3	-1	± 11

Proof of 5.3. Carry over the notation of §4 for $\mathrm{KO}(B)$. Let $\mathrm{KO}^*(HP^\infty) = \mathrm{KO}^*(pt)[[u]]$, and let $v = \eta_4 u$, $w = \eta_8 u^2$. By Lemma 4.6 we may choose u so that

$$\Psi^2 v = 4v + 2w\,.$$

We will compute $a(B)$ by chasing diagram 5.1 with KO-theory. Since M_0 and M_1 have very bad torsion, we must localize KO-theory at the relevant primes. So let

$$_{1/2}\mathrm{KO}^*(X) = \varprojlim_n \mathrm{KO}^*(X^n) \otimes Z[\tfrac{1}{2}]$$

$$_2\mathrm{KO}^*(X) = \varprojlim_n \mathrm{KO}^*(X^n) \otimes Z_{(2)}\,,$$

where $Z_{(2)}$ is the integers localized at the prime ideal (2). These theories are cohomology theories with ring and Ψ^k structures.

We may chose generators $x_0 \in {}_2\mathrm{KO}^4(M_0)$ and $x_1 \in {}_{1/2}\mathrm{KO}^4(M_1)$ so that $\rho_i^! x_i = x$, where we will use the symbol x also for the inclusion of x in ${}_{1/2}\mathrm{KO}$ or ${}_2\mathrm{KO}$. Then

$$ {}_2\mathrm{KO}(M_0) = {}_2\mathrm{KO}(pt)[[x_0]] $$

$$ {}_{1/2}\mathrm{KO}(M_1) = {}_{1/2}\mathrm{KO}(pt)[[x_1]] . $$

Let $y_i = \eta_4 x_i$, $z_i = \eta_8 x_i^2$, so

$$ \Psi^2 y_i \equiv 4y_i + 2az_i \pmod{\mathrm{KO}_9} . $$

Now

$$ f_i^! x_i = n_i u + e_i \eta_4 u^2 \pmod{\mathrm{KO}_9} $$

so

$$ f_i^! y_i = n_i v + 4e_i w . $$

Also

$$ \Psi^2 f_i^! y_i = 4n_i u + (2n_i + 64e_i)w $$

$$ f_i^! \Psi^2 y_i = 4n_i v + (2n_i^2 a + 16e_i)w . $$

So, recalling that the denominator of e_0 is prime to 2 and that e_i is a power of 2, we have

$$ a \equiv 1/n_0 \pmod 8 $$

$$ a \equiv 1/n_1 \pmod 3 . $$

But $1/n_0 \equiv n_0 \pmod 8$, $1/n_1 \equiv n_1 \pmod 3$ so

$$ a \equiv 9n_0 - 8n_1 \pmod{24} . $$

§ 6. The Hilton-Roitberg example

To illustrate how the techniques of §4 and §5 may be used to construct and study subgroups of exotic FD-groups, we will show that for each BS^3 constructed in §5, there is a multiplication on the Hilton-Roitberg "criminal" [18, 30] containing that BS^3. Associative multiplications on the Hilton-Roitberg bundle were first constructed by Stasheff [30].

We shall consider principal S^3 bundles over S^7. We will distinguish homotopy types of total spaces by a KO-theory invariant.

Let E be a simply connected complex such that $H^*(E;Z) = \Lambda_Z[t_3, t_7]$, dim $t_i = i$. The total space of an S^3 fibration over S^7 is such a space. Using the Atiyah-Hirzebruch spectral sequence,

$$\mathrm{KO}^*(E) = \Lambda_{\mathrm{KO}^*(pt)}[t_3, t_7] ,$$

where t_3 and t_7 denote representatives of the corresponding elements in $H^*(E;Z)$. We want to compute Ψ^2 in $\mathrm{KO}^{-1}(E) = \mathrm{KO}(SE)$. Put $u = \eta_4 t_3$, $w = \eta_8 t_7$, then

$$\Psi^2 u = 4u + 4bw \quad (\mathrm{mod}\ \mathrm{KO}_9) ,$$

using, as in §4, complexification and the fact that $\Psi^2 u \equiv u^2 \pmod 2$ [6, 3.2.2]. If t_3' is another choice for t_3, $t_3' = t_3 + e\eta_4 t_7$, then for $u' = \eta_4 t_3'$, $u' = u + 4ew$. As in §4 we compute

6.1. Lemma. *$\pm b(E)$ is a homotopy invariant (mod 12).*

Let $S^3 \to E \to S^7$ be a fibration, and let $f\colon S^7 \to S^7$ be of degree d. Let E_f be the fibre product

$$\begin{array}{ccc} S^3 & = & S^3 \\ \downarrow & & \downarrow \\ E_f & \to & E \\ \downarrow & & \downarrow \\ S^7 & \xrightarrow{f} & S^7 \end{array} .$$

Then an easy verification gives

6.2. Lemma. *$b(E_f) \equiv \pm \deg f \cdot b(E)$ (mod 12).*

Now let B be a simply connected complex with $H^*(B;Z) = Z[x_4, x_8]$, dim $x_i = i$ (e.g. BSp(2)). We have

$$\mathrm{KO}^*(B) = \mathrm{KO}^*(pt)\,[[x_4, x_8]] .$$

Put $u = \eta_4 x_4$, $v = \eta_8 x_4^2$, $w = \eta_8 x_8$. Using the Serre spectral sequence, we have

$$H^*(GB;Z) = \Lambda_Z[t_3, t_7] ,$$

where, under the adjoint map $SGB \to B$ and suspension, $x_4 \to t_3$, $x_8 \to t_7$. Using the same map, $u \to u$, $w \to w$ and

$$\psi^2 u = 4u + 2av + 4bw\ ,$$

where of course $\pm b$ is well determined (mod 12).

Suppose B contains an FD-subgroup $\tilde{B}S^3$, where $\tilde{B}S^3$ is some FD-group structure on S^3. Then we have a quotient fibration

$$S^3 \to GB \to S^7\ .$$

Now GB is an H-space; so, by a theorem of Curtis and Mislin [15], GB has the homotopy type of a principal S^3 bundle over S^7. All such are obtained as follows. Let $S^3 \to \mathrm{Sp}(2) \to S^7$ be the standard quotient of Lie groups. Let $f: S^7 \to S^7$ be of degree d and let E_d be the pull back bundle

$$\begin{array}{ccc} S^3 & = & S^3 \\ \downarrow & & \downarrow \\ E_d & \to & \mathrm{Sp}(2) \\ \downarrow & & \downarrow \\ S^7 & \to & S^7 \end{array}\ .$$

Now principal bundles over S^7 are classified by an element of $\pi_6 S^3 = Z_{12}$ with generator ω corresponding to $\mathrm{Sp}(2) \to S^7$. So E_d corresponds to $d\omega$. Using Lemma 6.7 below, $b(\mathrm{Sp}(2)) = \pm 1$ so

$$b(E_d) \equiv \pm d \pmod{12}\ .$$

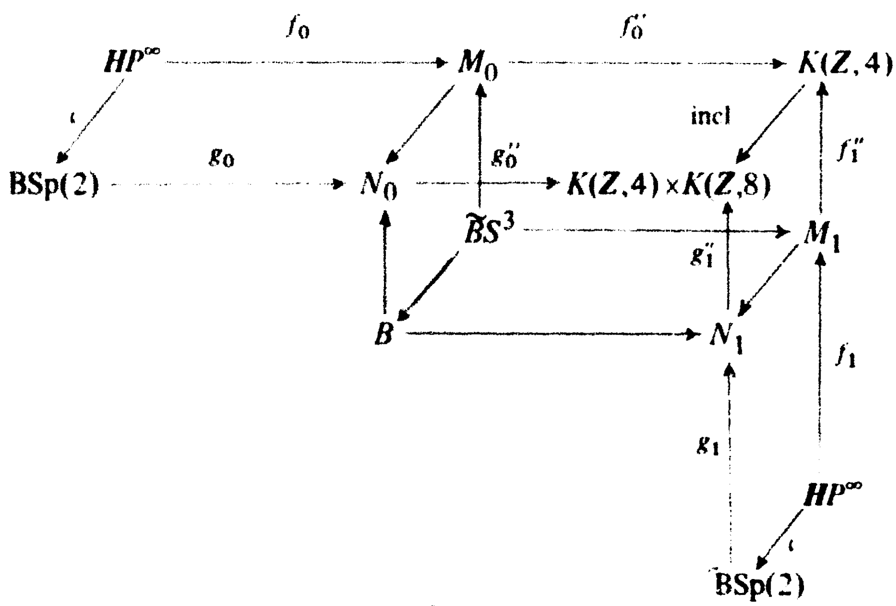
The number b thus classifies the homotopy type of the space E_d since $E_{-d} \simeq E_d$. We have proved

6.3. Proposition. *If B is an FD-group structure containing a group structure on S^3, and if $H^*(B; Z) = Z[x_4, x_8]$, then GB has the homotopy type of E_d where $d \equiv \pm b(B) \pmod{12}$.*

The Hilton-Roitberg criminal is $E_{\pm 5}$. We may now prove

6.4. Theorem. *Let $\tilde{B}S^3$ be one of the group structures on S^3 constructed in §5. Then there is a group structure on* Sp(2) *and a group structure on $E_{\pm 5}$ containing $\tilde{B}S^3$ as an FD-subgroup structure.*

Proof. Let BSp(2) be the standard (Lie) structure on Sp(2). We have an inclusion $HP^\infty \xrightarrow{\iota} \mathrm{BSp}(2)$. $H^*(\mathrm{BSp}(2); Z) = Z[x_4, x_8]$, $H^*(HP^\infty; Z) = Z[\iota^* x_4]$, and $\iota^* x_8 = 0$. As in §5 we may construct the following commutative diagram



where

(i) f_i, f_i'', n_i are as in §5,

(ii) g_i, g_i'' represents $n_i x_4 \oplus m_i x_8$, $i = 0, 1$,

(iii) m_0 prime to 2, m_1 prime to 3, 5, 7, ...,

(iv) g_0, g_1'' are 2 equivalences,

(v) g_1, g_0'' are 3, 5, 7, ... equivalences.

By Proposition 6.3, B is a group structure on E_d, $d = b(B)$, and it is easy to see that $\tilde{B}S^3$ is a FD-subgroup of B. It remains to calculate $b(B)$. But using Lemma 6.7 below and the technique of §5,

6.6. Lemma. $\pm b(B) \equiv 4m_1n_1 - 3m_0n_0$ *(mod 12).*

The theorem now follows from the table

$a(BS^3)$	n_0 (mod 4)	n_1 (mod 3)	m_0 (mod 4)	m_1 (mod 3)	b (mod 12)
±1	1	1	1	1	±1
±1	1	1	1	−1	±5
±5	−1	1	−1	1	±1
±5	−1	1	1	1	±5
±7	1	−1	1	−1	±1
±7	1	−1	1	1	±5
±11	−1	−1	1	1	±1
±11	−1	−1	1	−1	±5

6.5. Remark. It is not yet known whether there is a group structure on $E_{\pm 5}$ with a subtorus of maximal rank. One can, however, get an exotic "quaternionic torus" $\widetilde{B}S^3 \times \widetilde{B}S^3$ contained in a $BE_{\pm 5}$. One uses the $S^3 \times S^3 \to \mathrm{Sp}(2)$ and the above mixing procedure with m_0 and m_1 equal to 1. By comparing the table following 6.6 with that following 5.3, one sees that the $\widetilde{B}S^3$ which arise are those with $a \equiv \pm 5$ or ± 7 (mod 24).

6.7. Lemma. $b(\mathrm{BSp}(2)) = -1$.

Proof. Let T^2 be the maximal torus of $\mathrm{Sp}(2)$. $H^*(B_{T^2}; Z) = Z[x_1, x_2]$, $\dim x_i = 2$. Let $H^*(\mathrm{BSp}(2), Z) = Z[t_4, t_8]$ and, if ι is the inclusion of T^2, $\iota^* t_4 = x_1^2 + x_2^2$, $\iota^* t_8 = x_1^2 x_2^2$. We have the complex representation ring $\mathrm{RU}(T^2) = Z[z_1, z_1^{-1}, z_2, z_2^{-1}]$, and $\mathrm{KU}(B_T) = \widehat{\mathrm{RU}}(T^2)$, with $\mathrm{KU}(B_T) = Z[[\xi_1, \xi_2]]$, $\xi_1 = z_1 - 1$, $\xi_2 = z_2 - 1$, where ξ_i represents x_i. Now [2, 7.6] $\mathrm{RU}(\mathrm{Sp}(2)) = Z[\lambda_1, \lambda_2]$, where λ_i = are the ith symmetric functions of $z_1, z_1^{-1}, z_2, z_2^{-1}$. Furthermore, λ_1 is symplectic and λ_2 real. One calculates that

$$\eta_1 = (\lambda_1 - 4)$$
$$\eta_2 = (\lambda_2 - 2\lambda_1 + 2)$$

represent $\iota^* t_4 = x_1^2 + x_2^2$ and $\iota^* t_8 = x_1^2 x_2^2$ respectively. Furthermore, η_1 is symplectic and η_2 real. Calculate

$$\psi^2 \eta_1 = 4\eta_1 + \eta_1^2 - 2\eta_2 .$$

Transferring this information to KO-theory via the results of [4] gives the lemma.

Appendix

Fibre homotopy equivalence of simplicial fibrations

We wish to have in simplicial language the results of Dold and Lashoff [16]. Let X be a Kan complex. We have a complex X^X of maps of X into X. An n-simplex of X^X is a fibre map

$$\begin{tikzcd} X \times \Delta_n \arrow[r, "f"] \arrow[d] & X \times \Delta_n \arrow[d] \\ \Delta_n \arrow[r] & \Delta_n \end{tikzcd}$$

where Δ_n is the standard n-simplex. A theorem of Moore [e.g. 24, p.69] states the X^X is a Kan-complex.

A.1. Definition. The complex $H(X)$ of homotopy equivalences of X is the subcomplex of X^X of maps f above such that f restricts to a homotopy equivalence on the fibre over a vertex of Δ_n. Clearly

A.2. Lemma. *$H(X)$ is a simplicial monoid and a Kan complex.*

Moore [26] gives a construction of a principal classifying fibration

$$H(X) \to WH(X) \to \bar{W}H(X) = B_{H(X)}$$

so we may talk about bundles with fibre X and structure monoid $H(X)$. We wish to prove

A.3. Proposition. *Fibre homotopy classes of Kan fibrations*

$$X \to E \to B$$

with fibre X and base B are in one to one correspondence with homotopy classes of maps $B \to B_{H(X)}$.

Proof. We must show that every fibration is fibre homotopy equivalent to an $H(X)$ bundle and any $H(X)$ bundles which are fibre homotopy equivalent are equivalent. Let F,

$$F \xrightarrow{i} X \xrightarrow{j} F$$

be a minimal subcomplex, j a retraction, $ji = 1$, $ij \sim 1$. Then i and j induce a homomorphism

$$H(F) \to H(X)\,,$$

$$\alpha \to i \circ \alpha \circ j\,.$$

Since $i \circ j \sim 1$, this map is a homotopy equivalence. Therefore we have a homotopy equivalence

$$B_{H(F)} \to B_{H(X)}\,.$$

But clearly $H(F) = A(F)$ [8], the automorphism complex of F, since F is minimal. But $B_{A(F)}$ classifies isomorphism classes of minimal fibrations [8], and fibre homotopy equivalences of minimal fibrations are isomorphisms. Furthermore, every Kan fibration has a minimal fibration as a deformation retraction [26]. The proposition now follows.

One would like to relate this to the topological case. Let RX be the geometric realization of X. For any n-simplex $f\colon X \times \Delta_n \to X \times \Delta_n$ of X^X we can define a projection $\tilde{f}\colon X \times \Delta_n \to X$ and thus a map

$$R\tilde{f}\colon RX \times R\Delta_n \to RX\,.$$

But $R\Delta_n$ is the standard topological n-simplex so we have defined a singular simplex $R\widetilde{f} \in \operatorname{Sin} RX^{RX}$, where Sin denotes the singular complex. We thus have a simplicial map $H(X) \to \operatorname{Sin} H(RX)$, where $H(RX)$ is the space of homotopy equivalences of RX. There is an adjoint map

$$RH(X) \to H(RX)\,.$$

A folk conjecture is

A.3. Proposition. *The above map*

$$RH(X) \to H(RX)$$

is a weak homotopy equivalence.

The author intends to prove this in a future paper.

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